September 1964

BINOMIAL GROUP-TESTING WITH AN UNKNOWN PROPORTION OF DEFECTIVES*

by

Phyllis A. Groll and Milton Sobel
Stanford University University of Minnesota

Technical Report No. 48

* This research was supported in part by the National Science Foundation under Grant Number G-19126.

* Supported in part by the Army, Navy, and Air Force under Contract Nonr-225(53) (NR-042-002) with the Office of Naval Research.

Changes to

"Binomial Group-Testing with an Unknown Proportion of Defectives"

by Phyllis Groll and Milton Sobel

1. Title Page: Put asterisk after Stanford University and in footnote write "Now at Stanford Research Institute, Menlo Park, California."

2. Change last paragraph of Section 1.1 (bottom of page 2d) to read:

Finucan [6] recently made an interesting extension (and a correction) to Dorfman's work [4] but unfortunately he was not aware of any work since then on the group-testing problem. His extension still limits the experimenter to the following subclass of the set of all possible procedures. "He tests all his units in disjoint groups of equal (or approximately) equal size, say x_1, and this constitutes his first stage. He then tests all groups showing at least one defective (considering both cases: with and without recombining all these groups after the first stage) using disjoint subgroups of equal (or approximately equal) size, say $x_2 < x_1$, and this is his second stage. If he tests individual units the j^{th} time around then it is called a j-stage procedure." In attempting to compare his work with that in [8] and [9], we find that it is difficult to make numerical comparisons with his (or our) worked examples because he uses $p = 1/64$ and because of numerical misprints that were found. However if we substitute $r = 1/p = 100$ in his equation (7) for the minimum expected number of tests per unit classified (using the optimal number of stages), we obtain $c_{min} \approx (.02718)(4.60517) = .12517$. A comparable figure in Table Vc of [8] using $N = 100$ and $p = .01$ is .08320. In addition, in Table I of [9] for $p = .01$ (both for procedure R_{01} and R_{21}) the limiting value of the expected number of tests per unit classified as $N \to \infty$ is .08105, which is conjectured to be an optimal value. It should be noted that the ideas referred to as "refinements" in [6] are all incorporated in the basic recursion formulas in [8], [9] and the present paper. Finally, it is interesting to note that Finucan's extension is appropriate for and closely related to the procedure developed in [8] for the special application[1] described above; the only difference is that the latter procedure for this restricted problem does not use "stages" and hence does not assume equal test-group sizes within a stage.

Additions and Corrections to
"Binomial Group Testing with an Unknown Proportion of Defectives"

by Phyllis A. Groll and Milton Sobel

1. Add as a new paragraph at the bottom of Page 19:

It is interesting to note that Table III can be used to obtain an interval approximation of both $\bar{H}_{0,0}(N|R^{(1)})$ and $\bar{H}_{0,0}(N|\tilde{R}^{(0)})$ for any starting integer N. The first differences in $\bar{H}_{0,0}(N|R^{(1)})$ and $\bar{H}_{0,0}(N|\tilde{R}^{(0)})$ are erratic but do show a downward trend and indeed appear to approach the constant .787... given in (4.4). Assuming that this downward trend is maintained as N increases and that the above limit is correct (these have not been proved), it is then clear that we can use the last two entries in Table III to obtain for $\bar{H}_{0,0}(N|R^{(1)})$ the "approximate interval"

$$[32.225 + .787(N-36),\quad 32.225 + .876(N-36)] \qquad \text{for } N \geq 36,$$

and similarly for $\bar{H}_{0,0}(N|\tilde{R}^{(0)})$ the "approximate interval"

$$[21.699 + .787(N-24),\quad 21.699 + .878(N-24)] \qquad \text{for } N \geq 24.$$

For example, for $N = 100$ this gives for $\bar{H}_{0,0}(100|R^{(1)})$ the approximate interval [82.6, 88.3] and for $\bar{H}_{0,0}(100|\tilde{R}^{(0)})$ it gives the approximate interval [81.5, 88.4].

2. In Table III change $H_{0,0}(N|\tilde{R}^{(0)})$ to $\bar{H}_{0,0}(N|\tilde{R}^{(0)})$.

Abstract

The binomial group-testing problem is extended to the case in which the common probability p of a unit being defective is unknown. A Bayes "non-mixing" prodedure $R^{(1)}$ is derived and compared with other procedures, in particular with the corresponding procedure R_1 that requires the knowledge of p and with another procedure based on continually revising the maximum likelihood estimate of p; the latter is called an empirical Bayes solution. Finally, a Bayes procedure derived later in the paper allows "mixing" and this is conjectured to be the unrestricted Bayes solution; the improvements due to "mixing" are shown to be small in Table III. Several applications of the general problem of group-testing are discussed in the introduction and this virtually constitutes a general review of the subject. After the procedure $R^{(1)}$ is defined a detailed illustration is given in section 2.1 showing how to carry out this procedure with the use of Tables I and II. Lower bounds for the Bayes risk associated with any group testing procedure for unknown p are given in Table III; one of these bounds, based on information theory, is derived in section 4.

1. Introduction:

The problem of group-testing is concerned with classifying each of N given units into one of two disjoint categories which we call satisfactory and unsatisfactory (or simply, good and bad). The characteristic feature is that any number of units x can be tested simultaneously but the information obtained from a single test on x units, without any chance of error, is that either (i) all x are satisfactory, or (ii) at least one of the x units is unsatisfactory, but it is not known which ones or how many. The problem is to devise a sequential sampling scheme which minimizes the expected number of tests required to classify all of the N units as satisfactory or unsatisfactory. Some simple schemes have been proposed in [2],[7].

The principal model considered is that the N units represent independent Bernoulli random variables with probability q and p = 1-q of being satisfactory and unsatisfactory, respectively. In this paper the case of unknown q is studied; the case of known q is considered in [4], [5] and [6]. Reference [4] deals with a simplified procedure, denoted by R_1, which is highly efficient for all values of q but lacks optimality for q close to one; reference [6] deals with a slightly more involved procedure, denoted by $\tilde{R}_0$, which is conjectured to be optimal for all values of q ($0 \leq q \leq 1$). In this paper sections 2 and 6 deal with two different Bayes solutions to the group-testing problem when the a priori distribution $\lambda(q)$ is known; one (in section 2) corresponds to the simplified R_1 procedure and the other (in section 6) corresponds to the more efficient procedure $\tilde{R}_0$. Section 3 deals with a Maximum Likelihood Estimation (MLE) procedure of R_1-type and two other procedures of R_1-type. Some lower bounds on the expected number of tests for any procedure, based on unknown q, are computed in Section 4. In section 5 it is pointed out that if each unit has its own q value and these q values are independently distributed with the common density $d\lambda(q)$ then the Bayes solution is to treat it like a problem with known parameter $\bar{q} = \int_0^1 q d\lambda(q)$. If the a priori distribution $\lambda(q|c)$ is known except for the parameter c, then one can usually re-estimate c after each test and the MLE procedure of section 4 then takes on the role of an Empirical Bayes procedure for the group-testing problem.

1.1 Applications of Group-Testing

The first application of group-testing [4], [5] in the literature was to the problem of pooling blood samples in order to classify each one of a large group of people (e.g. soldiers in an Army unit) as to whether or not they have a particular disease (e.g., syphilis). Dorfman in [4] puts on the restriction that a fixed number of samples are put in each group-test until a group is reached that does not pass. Then, as a second restriction, each unit in that group is tested separately. Sterrett in [11] drops the second restriction and in its place he tests one-at-a-time only until a defective unit is found. Then the remaining units (if any) from that group are tested as a group and again if it does not pass he tests one-at-a-time until a defective unit is reached. Comparisons of this procedure with that of Dorfman are made in [11] and further comparisons of these and other procedures can be found in [8].

An interesting feature about the applications of group-testing is the variety of different fields in which they appear. In [8] many industrial applications are mentioned. For example, in making a "leak test" on a large number of gas-filled (say, with helium) electrical devices, we can test any number of units in a single test and the result of a test on x units is that either all x are good or at least 1 of the x is defective. We do not know as a result of this test alone how many of the x units or which ones are defective. Since the major cost in this case was the time required for the group-tests (each test took about ½ hour) we were interested in minimizing the expected number of tests; thus we have the basic group-testing problem. If we assume that with respect to the leak or non-leak condition the units are independently binomial random variables with the same unknown initial probability p of leaking then we are in the framework of the present paper.

Another application is in testing various electrical devices such as condensers, resistors, etc; the main idea can best be explained with the familiar Christmas Tree background. We assume that the x bulbs for the tree are all in series so that when we switch on the lights (or plug into the wall socket) we will know by the result that either all the x bulbs are good or at least one of the x is defective. We do not know as a result of this test alone how many or which ones are defective. Suppose we had shorter wires (of various sizes) for fewer bulbs and used these to find out exactly which ones are defective. The cost here is again the time required to classify all the x units and this is assumed to be proportional to the number of tests required. Then under the appropriate assumptions of independence and a common unknown initial probability p of being defective we again have a group-testing problem within the framework of the present paper.

Many interesting restrictions can be placed on these problems and some of these are treated in [8] and [9]. We now consider some these restrictions; each restriction corresponds to some further application.

1. Suppose we take 3c.c. of blood from each person in the first applications above and use 1 c.c. of it in any pooled sample. Then, if we apply the restriction that we will not take blood more than once from any individual, it follows that each person can be involved in at most 3 group-tests, i.e., if any individual is not classified after two group-tests then he must be tested individually on the third c.c. of his blood sample. This problem is briefly considered in [8] and a solution is given there but the numerical answers are not computed for large numbers of people.

2. Another restriction that arises in practice is to assume that the

units in any test-group are indistinguishable from each other and that no attempt at marking them (or keeping them separate) is made. This restricts ones knowledge of the past history of the individual units in any subgroup, while the history of the subgroups is known. The reason for this restriction is simply to avoid the cost of marking or otherwise keeping track of individual units. Some remarks on this are made in [8] and [9]; in particular, it is conjectured that for the case of known p (or q) and any finite initial population the so-called "non-mixing" procedure R_1 defined in [8] is optimal in the sense that it minimizes the expected number of tests. For the case of an infinite initial population a similar result holds but the definition of optimal (see [9]) in this case is necessarily different.

3. A third type of restriction could arise in the Christmas Tree problem above. Suppose the bulbs are kept in place and the experimenter can apply a voltage across any succession of adjacent bulbs (the group-test), but cannot test any arbitrary subset. This restricts the set of possible group-tests available to the experimenter since he can only test "intervals". We still wish to minimize the expected number of group-tests required to classify all the bulbs. This restriction is mentioned in [9] where it is conjectured that for this problem the non-mixing procedure R_1 (slightly modified by adding specific instructions to take the next test group from the left end of the defective set if $m \geq 2$ or from the left end of the binomial set if $m = 0$) is again optimal.

Another application of group-testing is to the design of group-screening experiments in [2] and [13]. In its simplest form there are two levels for each of several variables and one or more of k factors at (say) the higher level is "highly correlated" with (or causes) a substantial increase in the observed response. We can take any subset of the k factors at the upper level and if we observe the larger response then we know that one or more of the factors at the upper level is a "highly correlated" factor. The problem is to minimize the number of observations required (assuming no error at all in the observations) to classify each of the k factors as being "highly correlated" or "not highly correlated" with the (increased) response. Whenever we start an investigation with a large number of factors we are often forced to ask whether we can cut down the number of factors to a few more important factors; the group screening techniques have been devised as efficient preliminary experimental methods for picking out the important factors. Further references on the far-reaching role of screening in the design of experiments can be found in [13] and a correction to an error in [13] is discussed in [3].

It is also possible (and perhaps even more practical) to consider group-testing problems in which the group-test results are uncertain. For example, a cancer patient may have a combination of several different drugs injected into him simultaneously. Let us suppose that we have some basis for assuming that the drugs act independently (and that their side-effects are cumulative). Then a simplified model might assume that any observed improvement after the injection was due to at least one of the injected drugs. A more sophisticated model might allow for the possibility that the observation that there was an improvement was incorrect (the uncertain group-test result) or it might also allow for the possibility that the improvement was not due to any of the drugs in the injection.

Finally, a totally different application of group-testing is to the area of table construction and checking. Suppose we have a table of the function $f(x)$ whose r^{th} differences are equal (or very close to being equal) to a known constant c, in some given interval of x-values; let h denote the constant difference of x-values in the table. To check the table we form an arithmetic progression of $r+1$ values of x in the table. Let (x_1,x_2) denote the first and last value and let kh denote the constant difference of the x values in this progression. If the r^{th} difference for this progression is equal to (or close to) kc then we act as if there is no error between x_1 and x_2; otherwise we assume there is an error and we look for a "narrower" progression of $r+1$ values within the interval (x_1,x_2) in order to pinpoint some error more exactly. Each computation of an $r^{\underline{th}}$ difference is a group-test and we wish to minimize the "expected" number of group-tests required to eliminate the errors; here the word "expected" is not defined until we make some assumptions about the propagation of errors. Another application to the same area of constructing and checking tables was given in [1]. Here a function can be computed either by a recursion formula or by an explicit longer formula; the longer formula is used as a check and all the entries computed by recursion between two checks constitute a group test. Assuming that the computations are independent with a common known probability of error on each an explicit solution for this problem (how often to check?) is given in [1] in terms of the ratio of the times needed by the recursion formula and that needed by the longer formula. It should be noted that the description of the Sobel-Groll procedure on page 7 of [1] is incorrect.

The extension of the group-testing structure to problems in which each test result can have more than two outcomes is considered by Kumar [7]. In the case of three possible outcomes they are called good, mediocre and defective and we have a condition of "nested dominance" where the three possible outcomes are: "at least 1 defective", at least 1 mediocre and no defectives" and "all good". Interesting extensions of the known results for 2 outcomes are obtained. The general case of r such outcomes is also considered by Kumar.

Finally Finncan in [6] makes some interesting extensions (and a correction) to Dorfman's work [4] but unfortunately he was not aware of any other work since then on the problem. Because of numerical errors in his paper (e.g., the last calculation in example A clearly sums to $C = .159$ and not .169; also the author ends up in example A with a better result

.1741 - .03 = .1441 for a 3-stage procedure than for a 4-stage procedure, although he says that the latter is "best of all".) and because he uses $p = 1/64$, it is difficult to make numerical comparisons with his work. However if we substitute $r = 1/p = 100$ in equation (7) for his minimum expected number of tests per unit classified (using the optimal number of stages), we obtain $c_{min} = (.02718)(4.60517) = .12517$; the comparable figure in Table V C of [8] for $N = 100$ and $p = .01$ is .08320 and in Table I of [9] for $p = .01$ and for both procedures R_{01} and R_{21} the limiting value as $N \to \infty$ is .08105. Clearly the reason is that the author limits himself in [6] to a subclass of the set of all possible procedures. It should also be noted that the essential ideas of his refinements are all incorporated in the basic recursion formulas in [8], [9] and in the present paper.

2. The Bayes Solution $R^{(1)}$

The procedure R_1 given in [4] has the property that at any stage of the experiment the unclassified units need only be kept in two distinguishable sets, a "binomial" set and a "defective" set. The latter is known to contain at least one unsatisfactory unit; about the former we know only (from the original assumptions) that the Bernoulli variables are independent with common q. Under procedures R_1 and $R^{(1)}$, these two sets are never mixed together to form a new group for the next test; this explains why we shall sometimes refer to $R^{(1)}$ as being a non-mixing procedure or as being of R_1-type.

At any stage of experimentation (i.e., between any two tests), let s and u denote the number of units definitely classified as satisfactory and unsatisfactory, respectively; let m and n-m denote the sizes of the current defective set and binomial set, respectively. Let $\lambda(q)$ denote a completely known a priori distribution of q; the a posteriori density (element) is then given by

$$(2.1)\qquad d\lambda_{s,u,m}(q) = \begin{cases} \dfrac{q^s(1-q)^u d\lambda(q)}{B(s+1,u+1)} & \text{for } m = 0 \\[2ex] \dfrac{q^s(1-q)^u(1-q^m)d\lambda(q)}{C(s,u,m)} & \text{for } m \geq 2 \end{cases}$$

where $C(s,u,m)$ and $B(s+1,u+1)$ are defined by

$$(2.2)\qquad C(s,u,m) = B(s+1,u+1) - B(s+m+1,u+1),$$

$$(2.3)\qquad B(s+1,u+1) = B(s+1,u+1|\lambda) = \int_0^1 q^s(1-q)^u d\lambda(q);$$

thus $B(s+1,u+1)$ reduces to the usual complete Beta function when $\lambda(q)$ is the uniform distribution on [0,1]. In (2.1) the top expression is for m = 0 and this is called the H-situation; the bottom expression is for $m \geq 2$ and this is called the G-situation. The case m = 1 reduces immediately (without testing) to an H-situation, if we classify (as unsatisfactory) the one unit left in the defective set of size one.

The expected number of additional tests required at any stage depends on s and u as well as on m, n and q; let $G^{(i)}_{s,u}(m,n) = G_{s,u}(m,n|q,R^{(i)})$ denote the expected number of additional tests required when the procedure $R^{(i)}$ is used; for convenience we denote $G^{(i)}_{s,u}(0,n)$ by $H^{(i)}_{s,u}(n)$.

<u>Bayes Procedure $R^{(1)}$</u> (Non-mixing or R_1-type)

The basic recurrence relations which implicitly define the Bayes procedure $R^{(1)}$ are, for $m = 0$, $n \geq 1$, $s \geq 0$ and $u \geq 0$,

$$(2.4) \qquad H^{(1)}_{s,u}(n) = 1 + q^x H^{(1)}_{s+x,u}(n-x)+(1-q^x)G^{(1)}_{s,u}(x,n)$$

where x is the integer that minimizes the integral of the right side of (2.4) with respect to the a posteriori density $d\lambda_{s,u,0}(q)$; for $2 \leq m \leq n$, $s \geq 0$ and $u \geq 0$,

$$(2.5) \qquad G^{(1)}_{s,u}(m,n) = 1+\left(\frac{q^x-q^m}{1-q^m}\right) G^{(1)}_{s+x,u}(m-x,n-x)+\left(\frac{1-q^x}{1-q^m}\right) G^{(1)}_{s,u}(x,n)$$

where x is the integer that minimizes the integral of the right side of (2.5) with respect to the a posteriori density $d\lambda_{s,u,m}(q)$. The obvious boundary conditions are for all s,u,q

$$(2.6) \qquad H^{(1)}_{s,u}(0) = 0$$

$$(2.7) \qquad G^{(1)}_{s,u}(1,n) = H^{(1)}_{s,u+1}(n-1) \quad \text{for } n \geq 1 .$$

For each triple (n,s,u) [resp., quadruple (m,n,s,u)] the integer x that minimizes (2.4) [resp., (2.5)] is the size of the next group to be tested. For $m \geq 2$, the assumption is made here, as for R_1 in [4], that all x units are to be taken from the defective set of size m, so that we must have $1 \leq x \leq m-1$.

It is both useful and interesting to rewrite the recurrence formulas in a simpler form. From (5) we have

$$(2.8)\qquad q^x d\lambda_{s,u,0}(q) = \frac{B(s+x+1,u+1)}{B(s+1,u+1)}\, d\lambda_{s+x,u,0}(q)$$

$$(2.9)\qquad (1-q^x)d\lambda_{s,u,0}(q) = \frac{C(s,u,x)}{B(s+1,u+1)}\, d\lambda_{s,u,x}(q)\ .$$

If we define the constants (i.e., not depending on q)

$$(2.10)\qquad \overline{H}^{(1)}_{s,u}(n) = \int_0^1 H^{(1)}_{s,u}(n)d\lambda_{s,u,0}(q)$$

$$(2.11)\qquad \overline{G}^{(1)}_{s,u}(m,n) = \int_0^1 G^{(1)}_{s,u}(m,n)d\lambda_{s,u,m}(q)\ ,$$

then integrating both sides of (2.4) and (2.5) with respect to the a posteriori density element $d\lambda_{s,u,m}(q)$ gives for $m = 0$ and $m \geq 2$, respectively

$$(2.12)\qquad \overline{H}^{(1)}_{s,u}(n) = 1+\min_{1\leq x\leq n}\int_0^1 [q^x\, \overline{H}^{(1)}_{s+x,u}(n-x)+(1-q^x)\overline{G}^{(1)}_{s,u}(x,n)]d\lambda_{s,u,0}(q)$$

$$= 1+\min_{1\leq x\leq n}\left\{\frac{B(s+x+1,u+1)\overline{H}^{(1)}_{s+x,u}(n-x)+C(s,u,x)\overline{G}^{(1)}_{s,u}(x,n)}{B(s+1,u+1)}\right\}$$

$$(2.13)\qquad \overline{G}^{(1)}_{s,u}(m,n) = 1+\min_{1\leq x\leq m-1}\int_0^1\left[\left(\frac{q^x-q^m}{1-q^m}\right)\overline{G}^{(1)}_{s+x,u}(m-x,n-x)+\left(\frac{1-q^x}{1-q^m}\right)\overline{G}^{(1)}_{s,u}(x,n)\right]d\lambda_{s,u,m}(q)$$

$$= 1+\min_{1\leq x\leq m-1}\left\{\frac{C(s+x,u,m-x)\overline{G}^{(1)}_{s+x,u}(m-x,n-x)+C(s,u,x)\overline{G}^{(1)}_{s,u}(x,n)}{C(s,u,m)}\right\}\ .$$

The boundary conditions, as in (2.6) and (2.7) are for all s,u

$$(2.14)\qquad \overline{H}^{(1)}_{s,u}(0) = 0$$

$$(2.15)\qquad \overline{G}^{(1)}_{s,u}(1,n) = \overline{H}^{(1)}_{s,u+1}(n-1) \qquad \text{for } n \geq 1$$

Equations (2.12) through (2.15) are now explicit, i.e., without integrals, and can easily be iterated on a computer. This has been done for $m \leq n \leq 16$ and the uniform a priori density $d\lambda(q) = dq$ on $[0,1]$; some results are given in Table I for $n \leq 16$, $s \leq 16$ and $u \leq 2$.

Of course, one can also express $G_{s,u}^{(1)}(m,n)$ and $H_{s,u}^{(1)}(n)$ as functions of q using (2.4) through (2.7); typical results# are (using 3 decimal place accuracy)

$$(2.16) \qquad G_{1,0}^{(1)}(5,5) = (5+5q+8q^2+2q^3+q^4)/(1+q); \bar{G}_{1,0}^{(1)}(5,5) = 4.873,$$

$$(2.17) \qquad H_{0,0}^{(1)}(6) = 6+q-q^2+4q^3-8q^4-q^5+2q^6; \bar{H}_{0,0}^{(1)}(6) = 5.686 .$$

The polynomial results need not be unique since the Bayes solution may not be unique. (See Table IA, for example, where $x = 1$ or 2 or 4 for $H_{1,0}^{(1)}(4)$). However, the Bayes risk $\bar{H}_{s,u}^{(1)}(n)$ is the same for each Bayes solution. Also the form of $H_{s,u}^{(1)}(n)$ is a polynomial of degree at most n with integer coefficients.

Let $F_{s,u}^{(1)}(m) = F_{s,u}(m|q,R^{(1)})$ denote the expected number of group tests required to "break down" a defective set of size m and reach the next H-situation, when the Bayes procedure $R^{(1)}$ is used and s,u are as defined above. Then $F_{s,u}^{(1)}(m)$ does not depend on n. Since the procedure $R^{(1)}$ first breaks down the defective set until a single unsatisfactory unit is found, it is clear that we can write

$$(2.18) \qquad G_{s,u}^{(1)}(m,n) = F_{s,u}^{(1)}(m)+\left(\frac{p}{1-q^m}\right)\sum_{i=1}^{m} q^{i-1}H_{s+i-1,u+1}^{(1)}(n-i) .$$

In analogy with (2.10)and (2.11), we define

$$(2.19) \qquad \bar{F}_{s,u}^{(1)}(m) = \int_0^1 F_{s,u}(m)d\lambda_{s,u,m}(q) ,$$

so that (2.18) after integration takes the form

---

for the uniform a priori density

$$(2.20) \qquad \overline{G}^{(1)}_{s,u}(m,n) = \overline{F}^{(1)}_{s,u}(m)+\frac{1}{C(s,u,m)}\sum_{j=s}^{s+m-1} B(j,u+1)\overline{H}^{(1)}_{j,u+1}(n+s-1-j) .$$

We can now prove the following useful result which also holds for procedure R_1.

<u>Theorem:</u> For any G(m,n)-situation ($n \geq m \geq 2$), the size of the next group-test under procedure $R^{(1)}$ does not depend on n.

<u>Proof:</u> Substituting the right side of (2.20) for all three $\overline{G}$'s in the two extreme members of (2.13), we find that the three summations cancel and we obtain the simpler recursion formula

$$(2.21) \qquad \overline{F}^{(1)}_{s,u}(m) = 1 + \frac{\underset{1\leq x\leq m-1}{\text{Min}}\left\{C(s+x,u,m-x)\overline{F}^{(1)}_{s+x,u}(m-x)+C(s,u,x)\overline{F}^{(1)}_{s,u}(x)\right\}}{C(s,u,m)} ,$$

which does not depend on n. Also, the boundary condition $\overline{F}^{(1)}_{s,u}(1) = F^{(1)}_{s,u}(1) = 0$ does not depend on n. Moreover, it is clear that (2.21), which does not depend on n, must define the same integer values $x = x_G(m)$, where the minimum is attained, as (2.13) and this proves the theorem.

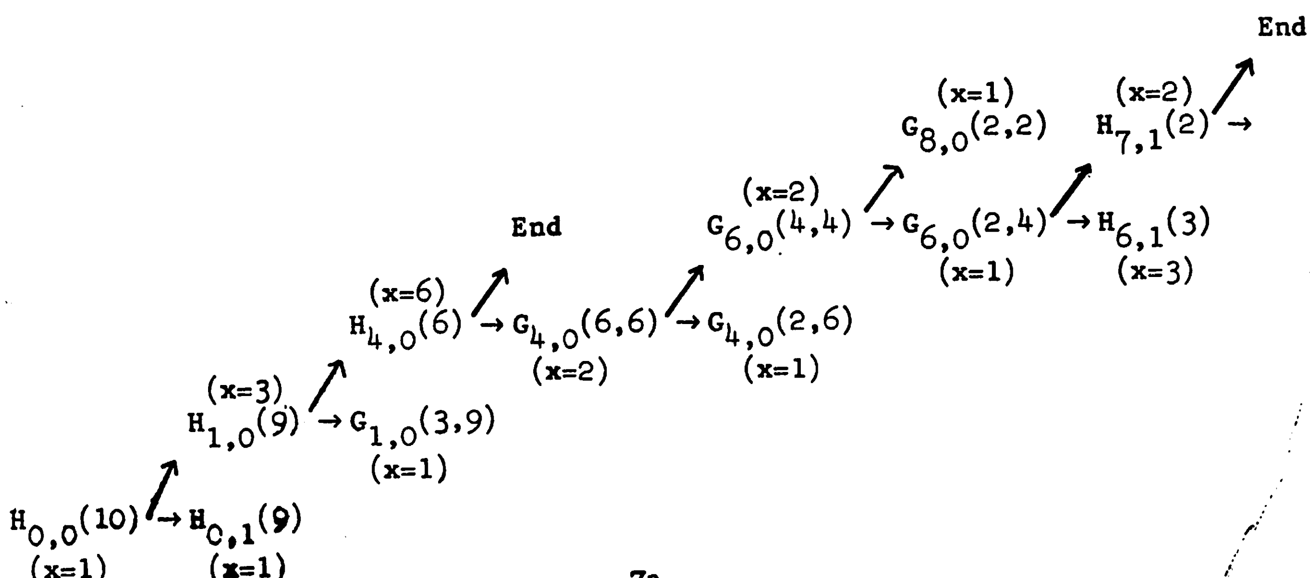
Thus to describe the procedure $R^{(1)}$ we need only give the x-value for each triple (s,u,m) in a G-situation and the x-value for each triple (s,u,n) in an H-situation; these x-values given in Table I define the procedure $R^{(1)}$.

In Table I, as in Figure 3 of [4], we note that the procedure appears to be approaching a limit as $n \to \infty$ in the sense that, for fixed u and x, the line separating x and x+1 tends to settle down near a particular value of s. For example, one is tempted to conjecture on the basis of the empirical results in Table IA that for u = 0 the limiting ($n \to \infty$) procedure of $R^{(1)}$ is to take x = s+1. It would be of interest to describe this limiting procedure for all s, u and x, since it would be an efficient procedure to use whenever the initial number of units N is large; this has not yet been done.

2.1 Illustration of Procedure $R^{(1)}$ with N = 10 and the Uniform U(0,1) A Priori.

To carry out procedure $R^{(1)}$ we use Tables I and II at the end of this paper which require that $\lambda(q)$ be a beta distribution with integer parameters. Suppose $\lambda(q)$ is the uniform $U(0,1)$ distribution and there are at the outset $N = 10$ units to classify. At the outset $s = u = 0$ and by Table I with $n = 10$ we start by testing one unit; suppose it is good. Then by Table I with $u = 0$, $s = 1$, and $n = 9$, we test 3 of the 9 remaining units; suppose they are also good. By Table I with $u = 0$, $s = 4$, and $n = 6$ we then test all 6 remaining units; suppose this test does not pass. Then switching over to Table II with $u = 0$, $s = 4$, and $m = 6$, we test 2 of the 6 unclassified units; suppose they are good. Using Table II again, with $u = 0$, $s = 6$ and $m = 4$, we now test 2 of the 4 units remaining to be classified; suppose they do not pass. Then by Table II again, with $u = 0$, $s = 6$ and $m = 2$, we test 1 of the two units in the defective set; suppose it is good. At this point we can classify 1 unit as defective by inference without a test and we are then back in an H(or binomial) situtation. Switching back to Table I with $u = 1$, $s = 7$ and $n = 2$, we then test the two remaining units; suppose they are good. Then we are through and it has taken us 7 tests to classify the 10 units; 9 good units and one defective unit were found.

<u>Diagram 1</u>: Incomplete Tree for Procedure $R^{(1)}$ with N = 10 and $\lambda(q) = U(0,1)$, Showing Two Sample Paths (or Branches).



The above details for carrying out procedure $R^{(1)}$ are included in Diagram 1, which shows two possible branches that might be obtained for $N = 10$ and $\lambda(q) = U(0,1)$; the complete tree for any N up to $N = 32$ can be obtained from Tables I and II but even for $N < 10$ it would take a great deal of space to write out a complete tree.

By Table III the expected number of tests $\overline{H}_{0,0}(N|R^{(1)})$ for $N = 10$ and $\lambda(q) = U(0,1)$ is 9.306. The more complicated Bayes procedure $\tilde{R}^{(0)}$ described in Section 6 below, which allows "mixing", is exactly the same for $N = 10$ as procedure $R^{(1)}$ and hence they have the same Bayes risk or average expected number of tests, i.e., $\overline{H}_{0,0}(10|R^{(1)}) = \overline{H}_{0,0}(10|\tilde{R}^{(0)})$. Although two lower bounds, 8.696 and 7.873, for $\overline{H}_{0,0}(10|R)$ are given in Table III for any group-testing procedure R, these lower bounds are in general not attainable and it is conjectured in Section 6 that $\tilde{R}^{(0)}$ is optimal and hence that 9.306 is the smallest possible Bayes risk for any group-testing procedure for $N = 10$ and $\lambda(q) = U(0,1)$. One of these lower bounds is derived in Section 4.

The interesting intuitive feature of the Bayes procedure $R^{(1)}$ that should be noted is that we increase (or maintain the same) group-test size as more good units are found and conversely the group-test size approaches one as more defective units are found; this property is more clearly seen in Tables I and II below.

For the case $N = 6$ the risk function or expected number of tests $H(6|R^{(1)})$ as a function of q has been computed and graphed in Figure 1 below. On the same graph we have also plotted the risk function of another procedure $R^{(2)}$ defined in Section 3 and in addition the expected number of tests for the procedure R_1 which requires the knowledge of q. Thus by comparing these functions for $R^{(1)}$ and R_1 we can assess the price we have to pay for not knowing the value of q but only its a priori distribution.

3. Other Procedures (Non-Mixing or R_1-type)

Another procedure $R^{(2)}$, comparable in some aspects with $R^{(1)}$, is based on the maximum likelihood estimate (m.l.e.) $\hat{q}$ obtained by maximizing the a posteriori density (2.1). Whenever the modal value of the a priori distribution is not unique we shall use some particular average of the modal values; for example, if the modal values form an interval then we arbitrarily take the midpoint of this interval as the m.l. estimate $\hat{q}$.

Procedure $R^{(2)}$: "At each stage of group-testing compute the value $\hat{q}$ of q that maximizes (2.1) and then follow the directions of procedure R_1 in [4], acting as if $\hat{q}$ were the true value of q."

This procedure makes use of the tables constructed for the best R_1-type procedure for known q in [4]; with the help of these and an auxiliary table giving m.l. estimates as a function of s,u and m, this procedure can be carried out easily with any initial number of units N. Such a table has been prepared for the case of an a priori density which is a Beta distribution $Cx^a(1-x)^b$ where a and b are non-negative integers. (See [In particular, if a = b = 1 then this gives the uniform a priori which is of special interest in the computations.] For m = 0, the estimate $\hat{q} = s_1/(s_1+u_1)$ where $s_1 = s + a$ and $u_1 = u + b$; for $m \geq 2$ the estimate $\hat{q}$ is the unique root between zero and one of

$$(3.1) \qquad \frac{d}{dq} \{q^{s_1}(1-q)^{u_1}(1-q^m)\}$$

or, letting $S = s_1 + u_1 + m$, the unique root between zero and one of

$$(3.2) \qquad s_1 - u_1 \sum_{i=1}^{m-1} \hat{q}^i - S\hat{q}^m = 0 .$$

To show this uniqueness, we first note that for $s_1 = 0$ the root is zero. For $s_1 \geq 1$, the left member of (3.2) changes sign at least once in the unit interval. On the other hand, by Descartes' Rule of Signs, it changes sign at most once on the positive $\hat{q}$ axis and hence at most once in the unit interval. Q.E.D.

The procedure $R^{(2)}$ has the strange property that if $a \geq 1$ and $b = 0$ then our initial $\hat{q} = 1$ and all the units are put into the 1st group test. Similarly, if $a = b = 0$ then our initial $\hat{q} = \frac{1}{2}$ (by an above assumption), and the first test is on one unit; if this unit is good then our new $\hat{q} = 1$ and all the remaining units are put into the 2nd group test. This means that $R^{(2)}$ cannot be used for the "assembly line" case, i.e., for $N = \infty$.

If the prior distribution λ does not assign positive measure to the point $q = 1$ and if the associated density is positive at the true value of q then, with probability 1, the two procedures $R^{(1)}$ and $R^{(2)}$ will both approach R_1 in the sense that for $q < 1$

$$(3.3) \qquad \lim_{N\to\infty} \frac{H_{0,0}^{(i)}(N)}{H_1(N)} = 1 \qquad (i = 1,2),$$

and the same relation holds with $H_{0,0}^{(i)}(N)$ replaced by $\overline{H}_{0,0}^{(i)}(N)$ $(i = 1,2)$. For $q = 1$, the two procedures, $R^{(1)}$ and $R^{(2)}$, get further and further apart as $N \to \infty$ since for $q = 1$ and the uniform a priori λ, $H_{0,0}^{(2)}(N) = 2$ for all $N \geq 2$ and $H_{0,0}^{(1)}(N) \to \infty$ as $N \to \infty$. Some graphical comparisons of $R^{(1)}$, $R^{(2)}$ and R_1 for $N = 6$ are given in Fig. 1 and some numerical comparisons are given in Table III.

A third procedure $R^{(3)}$ of the Non-mixing or R_1-type is based on the average information $\overline{I}(x) = \overline{I}(x;m,n,s,u)$ defined by

$$(3.4) \qquad \overline{I}(x) = \begin{cases} \int_0^1 D(q^x)\,d\lambda_{s,u,0}(q) & \text{for } m = 0 \\ \int_0^1 D\left(\frac{q^x - q^m}{1-q^m}\right)d\lambda_{s,u,m}(q) & \text{for } m \geq 2 \end{cases}$$

where

(3.5) $D(y) = -[y \log y + (1-y) \log (1-y)]$ for $0 \leqq y \leqq 1$

and x is an integer with $1 \leqq x \leqq n$ for $m = 0$ and $1 \leqq x \leqq m-1$ for $m \geqq 2$. The procedure $R^{(3)}$ is now defined by taking any integer x than maximizes first (second) expression for $\bar{I}(x)$ in (3.4) if $m = 0$ (if $m \geqq 2$). The quantity $\bar{I}(x)$ in (3.4) can also be regarded as the amount by which the entropy in our system of units is reduced if the next test contains x units; the top (bottom) expression applies if we start with an H-situation (G-situation). In a joint effort with I. Olkin, we have found an approximation to $R^{(3)}$ when $\lambda(q)$ is a polynomial in q by assuming that x is a continuous variable and differentiating under the integral sign in (3.4) and making use of tables of Laplace transforms.

A fourth procedure $R^{(4)}$ of the Non-mixing or R_1-type is a "Halving Procedure" that can be applied both for q known and q unknown. Actually, in [4], two halving procedures are considered, one with recombination of units with the same a posteriori distribution, the other without; we consider only the former one which is superior to the latter for all q. We now define $R^{(4)}$ as follows:

"Start by testing all N units. If this test fails, take the largest integer equal to or less than N/2 and test that many selected at random. At each step continue with the defective half-set until a single unsatisfactory unit is found. Then start all over again by testing the whole set of unclassified units (which has a binomial a posteriori distribution), and continue this until all units are classified."

This procedure is investigated in Appendix C of [4] and some numerical comparisons are given in Table II of [4]. It gives reasonable results if q is very close to unity, poor results if $q < 9$ and very poor results if

$q < 5$. It is included for comparison because it has a strong intuitive appeal to experimenters who are in a position to apply this procedure.

4. Lower Bounds for Any Group-Testing Procedure

Using the same methods described in [4] and [5] that are based on information theory and coding theory, it is possible to obtain lower bounds on $\overline{H}_{0,0}(N|R)$ for any (group-testing) procedure R. It is shown by Ungar [8] that for $0 \leq q \leq q_0 \equiv (\sqrt{5} - 1)/2$ the optimal procedure is to test units one at a time and it is pointed out in [4] that for any group-testing procedure R

$$(4.1) \qquad H(N|R) \geq -N[p \log p + q \log q] \quad ,$$

where the logs are taken to the base 2. Combining these, we have the stronger bound for any group-testing procedure R

$$(4.2) \qquad H_{0,0}(N|R) \geq \begin{cases} N & \text{for } 0 \leq q \leq q_0 \\ ND(q) & \text{for } q_0 \leq q \leq 1 \end{cases} \quad .$$

Integrating with respect to $d\lambda(q)$ gives for any group-testing procedure R

$$(4.3) \qquad \overline{H}_{0,0}(N|R) \geq N[\lambda(q_0^+) + \int_{q_0^+}^{1} D(q)d\lambda(q)]$$

where q_0^+ indicates an approach to q_0 from the right. If $\lambda(q)$ is the uniform distribution on [0, 1] then a calculation (to 3 decimal place accuracy) yields (.863) N for the right side of (4.3); some appropriate comparisons with these numbers are given in Table III.

5. Another Model

If we assume that each unit has its own q-value and that these q-values are independent chance variables with a common completely known density $d\lambda(q)$, then no new problem arises. A similar statement can be made when the common density $d\lambda(q|c)$ has an unknown scalar or vector parameter c.

The individual units are satisfactory with probability $\bar{q} = \int_0^1 q d\lambda(q)$ (which is known if λ is completely known) and unsatisfactory with probability $\bar{p} = 1-\bar{q}$; they are also mutually independent. Hence, the problem is reduced to the problem with known q (now replaced by $\bar{q}$) that is treated in [4] and the procedure R_1 can be applied.

Suppose now that $d\lambda(q|c)$ has an unknown parameter c, which can be estimated by an m.l. estimate $\hat{c}$ at each stage of group testing. Then $d\lambda(q|\hat{c})$ is the m.l. estimate of $d\lambda(q|c)$. The m.l. procedure $R^{(2)}$ can now be applied: "At every stage we insert a revised estimate $d\lambda(q|\hat{c})$, compute $\bar{q} = \int_0^1 q d\lambda(q|\hat{c})$ and then follow the directions of procedure R_1 in [4], acting as if $\bar{q}$ were the true value." This solution which revises the estimate of the a priori density of each stage can be regarded as an Empirical Bayes procedure for the group-testing problem.

For example, it may be that $c = (a,b)$ is a vector with a and b positive and (letting $B(a+1,b+1)$ denote the usual complete Beta function)

$$(5.1) \qquad d\lambda(q|c) = \frac{q^a(1-q)^b}{B(a+1,b+1)}\, dq \qquad (0 \leq q \leq 1)\ .$$

Then, if c is known, we use R_1 with

$$(5.2) \qquad \bar{q} = \int_0^1 \frac{q^{a+1}(1-q)^b}{B(a+1,b+1)}\, dq = \frac{B(a+2,b+1)}{B(a+1,b+1)} = \frac{a+1}{a+b+2}\ .$$

If c is unknown, then for any m we try to find the pair (a,b) that maximizes the averaged likelihood, i.e., the likelihood integrated with respect to the joint a priori density

$$(5.3) \qquad d\lambda(\vec{q}) = \prod_{i=1}^{n} \frac{q_i^a(1-q_i)^b dq_i}{B(a+1,b+1)} \quad .$$

The integrated likelihood $\bar{L}(a,b) = \bar{L}(a,b|s,u,m)$ expressed in terms of $\underline{a} = (a+1)/(a+b+2)$ and $\underline{b} = 1-\underline{a}$ is easily seen to be

$$(5.4) \qquad \bar{L}(a,b) = \begin{cases} C_1 \ \underline{a}^s \underline{b}^u & \text{for } m = 0 \\ C_2 \ \underline{a}^s \underline{b}^u (1-\underline{a}^m) & \text{for } m \geq 2 \end{cases}$$

If we take logs and set the partial derivatives with respect to a and b equal to zero then we find that we can only estimate ratios like $\underline{a}$ and $\underline{b}$. For $m = 0$ we get $s/(s+u)$ as an estimate of $\underline{a}$; for $m \geq 2$ we get the same equation as (3.2) with $\hat{q}$ replaced by $\underline{a}$. Thus the same table that gives estimates of q for procedure $R^{(2)}$ can be used here and in this sense the procedure is equivalent to $R^{(2)}$.

6. Bayes Procedure $\tilde{R}^{(0)}$: (Allows Mixing)

We now return to the original model of Sections 1 through 4 and consider a somewhat more complicated Bayes solution, in which the restriction to non-mixing or R_1-type procedures is removed. This solution is based on a new procedure $\tilde{R}_0$ described in [6], which is conjectured to be optimal for any known value of q; the construction of $\tilde{R}^{(0)}$ from $\tilde{R}_0$ assures us that if $\tilde{R}_0$ is optimal for the problem with known q then $\tilde{R}^{(0)}$ is optimal for the problem with unknown q. Hence, it is conjectured that $\tilde{R}^{(0)}$ is an optimal Bayes solution.

To describe the solution, we first need some preliminary definitions. It is shown in [6] (see also [5]) that, for a $G(m,n)$ situation with $m \geq 4$, the optimal solution is to take the next group(for testing) all from the defective set, without mixing. Hence, we need only study the "mixing routines" for $m = 2$ and $m = 3$. As in [5] and [6], if $\vec{a} = (a_1,\ldots,a_m)$ and $\vec{b} = (b_1,\ldots,b_{n-m})$ denote the set of units in the defective set and binomial set, respectively, then the mixing routine for the $G_{s,u}(2,n)$-situation with $n \geq 3$ is described by

$$(6.1) \qquad G_{s,u}(2,n): \quad \begin{tikzcd} & S_{s+n-1,u+1} & S_{s+n-1,u+1} \\ T(a_1,\vec{b}) \arrow[ur, "p"] \arrow[r, "f"'] & T(a_2,\vec{b}) \arrow[ur, "p"] \arrow[r, "f"'] & J_{s,u}(2,n) \end{tikzcd} \quad ,$$

where T( ) denotes a test on the units shown in the parens, f and p denote the courses to follow if a test fails or passes, respectively, $S_{s,u}$ denotes a terminal stop with s satisfactory and u unsatisfactory units, and $J_{s,u}(2,n)$ denotes a new a posteriori situation. For $n = 2$, it is clear that we must replace $J_{s,u}(2,n)$ in (6.1) by $S_{s,u+2}$. Similarly, for the $G_{s,u}(3,n)$ situation, the mixing routine is

$$(6.2) \quad G_{s,u}(3,n): \quad \begin{tikzcd} & S_{s+n-1,u+1} & S_{s+n-1,u+1} & S_{s+n-1,u+1} \\ T(a_1,a_2,\vec{b}) \arrow[ur, "p"] \arrow[r, "f"'] & T(a_1,a_3,\vec{b}) \arrow[ur, "p"] \arrow[r, "f"'] & T(a_2,a_3,\vec{b}) \arrow[ur, "p"] \arrow[r, "f"'] & J_{s,u}(3,n) \end{tikzcd} \quad ,$$

where $J_{s,u}(3,n)$ is again a new a posteriori situation. In both cases,

$m = 2$ and 3, the mixing routine is not always followed; it turns out that the alternative in both cases is to test a single unit from the defective set, without mixing. Thus the basic structure of the procedure $\tilde{R}^{(0)}$ will be similar to $R^{(1)}$, the major difference being that we have to show how to "break down" the new $J(2,n)$ and $J(3,n)$ situations. These details are given in [6] and also in Table V of this paper.

For convenience, let

$$(6.3) \qquad D_2(s,u,n) = C(s,u+1,n-1) + C(s+1,u+1,n-2)$$

where $C(x,y,z)$ is defined in (2.2) and let

$$(6.4) \qquad D_3(s,u,n) = C(s,u+1,n-1) + C(s+1,u+1,n-2) + C(s+2,u+1,n-3).$$

$^{\#}$Let $G_{s,u}^{(0)}(m,n)$, $H_{s,u}^{(0)}(n)$, $\bar{G}_{s,u}^{(0)}(m,n)$ and $\bar{H}_{s,u}^{(0)}(n)$ be defined for procedure $\tilde{R}^{(0)}$ exactly as was done for procedure $R^{(1)}$ in Section 2. Let $\bar{J}_{s,u}^{(0)}(2,n)$ for $n \geq 2$ and $\bar{J}_{s,u}^{(0)}(3,n)$ for $n \geq 3$ be defined by

$$(6.5) \qquad J_{s,u}^{(0)}(2,n) = \int_0^1 \frac{J_{s,u}^{(0)}(2,n)\; q^s(1-q)^u[(1-q^{n-1})+q(1-q^{n-2})]}{D_2(s,u,n)}\, d\lambda(q)$$

$$(6.6) \qquad \bar{J}_{s,u}^{(0)}(3,n) = \int_0^1 \frac{J_{s,u}^{(0)}(3,n)\; q^s(1-q)^u[(1-q^{n-2})+q(1-q^{n-3})+q^2(1-q^{n-4})]}{D_3(s,u,n)}\, d\lambda(q) ,$$

where $J_{s,u}^{(0)}(2,n)$ and $J_{s,u}^{(0)}(3,n)$ are the expected number of tests required by procedure $\tilde{R}^{(0)}$ if we start with the $J(2,n)$ and $J(3,n)$ situations defined by (6.1) and (6.2), respectively, with s and u units previously classified as satisfactory and unsatisfactory, respectively. For $n = 2$, we have $J_{s,u}^{(0)}(2,2) = \bar{J}_{s,u}^{(0)}(2,2) = 0$ for all $s \geq 0$, $u \geq 0$.

The basic recurrence formulas for $\tilde{R}^{(0)}$ are in part quite similar to those of $R^{(1)}$ in (2.12) and (2.13), except that the second one is used only for $m \geq 4$ and we need new equations for $m = 2$ and $m = 3$. After integration,

$\#$ For $n=3$ the last term in (6.4) vanishes and $D_2(s,u,3) = D_3(s,u,3)$.

the results are for $n \geq 1$, $s \geq 0$ and $u \geq 0$

$$(6.7)\quad \bar{H}^{(0)}_{s,u}(n) = 1 + \min_{1\leq x\leq n} \left\{ \frac{B(s+x+1,u+1)\bar{H}^{(0)}_{s+x,u}(n-x)+C(s,u,x)\bar{G}^{(0)}_{s,u}(x,n)}{B(s+1,u+1)} \right\},$$

and for $n \geq m \geq 4$, $s \geq 0$ and $u \geq 0$

$$(6.8)\quad \bar{G}^{(0)}_{s,u}(m,n) = 1 + \min_{1\leq x\leq m-1} \left\{ \frac{C(s+x,u,m-x)\bar{G}^{(0)}_{s+x,u}(m-x,n-x)+C(s,u,x)\bar{G}^{(0)}_{s,u}(x,n)}{C(s,u,m)} \right\}$$

For $m = 2$ (resp., $m = 3$) it turns out that under procedure $\tilde{R}^{(0)}$ in a $G(2,n)$ (resp., $G(3,n)$) situation we either take one unit from the defective set or follow the mixing routine in (6.1) (resp., (6.2)); this gives us the two equations,

$$(6.9)\quad \bar{G}^{(0)}_{s,u}(2,n) = \min \left\{ 1 + \frac{B(s+1,u+2)\bar{H}^{(0)}_{s,u+1}(n-1)+B(s+2,u+2)\bar{H}^{(0)}_{s+1,u+1}(n-2)}{C(s,u,2)}, \right.$$
$$\left. \frac{D_2(s,u,n)\bar{J}^{(0)}_{s,u}(2,n)+2B(s+1,u+2)+2B(s+2,u+2)-B(s+n,u+2)}{C(s,u,2)} \right\}$$

$$(6.10)\quad \bar{G}^{(0)}_{s,u}(3,n) = \min \left\{ 1 + \frac{B(s+1,u+2)\bar{H}^{(0)}_{s,u+1}(n-1)+C(s+1,u,2)\bar{G}^{(0)}_{s+1,u}(2,n-1)}{C(s,u,3)}, \right.$$
$$\left. \frac{D_3(s,u,n)\bar{J}^{(0)}_{s,u}(3,n)+\sum_{\alpha=1}^{3} B(s+\alpha,u+2)-3B(s+n,u+2)}{C(s,u,3)} \right\}.$$

We now consider the breakdown of the $J(2,n)$ and $J(3,n)$ situations separately, since there are a number of (different) equations needed in each case; all of these are obtained by integrating the corresponding equations in [6]. The procedures corresponding to these equations are given in Table V.

The following three equations all deal with $m = 2$. If $n = 3$ or 4,

$$(6.11)\qquad \bar{J}^{(0)}_{s,u}(2,n) = 1 + \frac{C(s,u+1,2)\bar{G}^{(0)}_{s,u+1}(2,n-1)+D_2(s+1,u,n-1)\bar{J}^{(0)}_{s+1,u}(2,n-1)}{D_2(s,u,n)} .$$

If $n \geq 5$ and not of the form $2^r + 2$ for any integer r,

$$(6.12)\qquad \bar{J}^{(0)}_{s,u}(2,n) = 1 + \frac{C(s+1,u+1,n-2)\bar{G}^{(0)}_{s+1,u+1}(n-2,n-2)+C(s,u+1,n-1)\bar{G}^{(0)}_{s,u+1}(n-1,n-1)}{D_2(s,u,n)} .$$

If $n = 2^r + 2$ for some integer $r \geq 2$ (letting $t = 2^{r-1} \geq 2$),

$$(6.13)\qquad J^{(0)}_{s,u}(2,n) = 1 + \frac{C(s,u+1,t)[1+\bar{G}^{(0)}_{s,u+1}(t,n-1)]}{D_2(s,u,n)}$$

$$+ \frac{C(s+1,u+1,t)[1+\bar{G}^{(0)}_{s+1,u+1}(t,n-2)]+D_2(s+t,u,z+2)\bar{J}^{(0)}_{s+t,u}(2,t+2)}{D_2(s,u,n)} .$$

The following five equations all deal with $m = 3$.# If $n = 4$ or 5,

$$(6.14)\qquad \bar{J}^{(0)}_{s,u}(3,n) = 1+\frac{C(s,u+1,n-1)\bar{G}^{(0)}_{s,u+1}(n-1,n-1)+D_2(s+1,u,n-1)\bar{J}^{(0)}_{s+1,u}(2,n-1)}{D_3(s,u,n)} .$$

If $n = 2^r + 2$ for some integer $r \geq 2$ (letting $t = 2^{r-1}$)

$$(6.15)\qquad \bar{J}^{(0)}_{s,u}(3,n) = 1+\frac{(s+2)D_3(s,u,t+1)+D_3(s+z,u,t+2)\bar{J}^{(0)}_{s+t,u}(3,t+2)}{D_3(s,u,n)}$$

$$+ \sum_{i=1}^{t} \frac{C(s+i-1,u+1,3)}{D_3(s,u,n)}\ \bar{G}^{(0)}_{s+i-1,u+1}(3,n-i) .$$

If $n = 7$, 8 or 9

$$(6.16)\qquad D_3(s,u,n)\bar{J}^{(0)}_{s,u}(3,n) = 2C(s,u+1,3)+C(s,u+1,n-1)+3C(s+1,u+1,n-2)$$

$$+ D_2(s+2,u,n-2)\bar{J}^{(0)}_{s+2,u}(2,n-2)+C(s+2,u+1,n-3)\bar{G}^{(0)}_{s+2,u+1}(n-3,n-3)$$

$$+ C(s+1,u+1,2)\bar{G}^{(0)}_{s+1,u+1}(2,n-2)+C(s,u+1,2)\bar{G}_{s,u+1}(2,n-1) .$$

If $2^r + 3 \leq n \leq 2^{r+1}$ for some integer $r \geq 3$ (letting $w = 2^{r-2}$)

If $n = 3$ we use the identity $\bar{J}^{(0)}_{s,u}(2,3) \equiv \bar{J}^{(0)}_{s,u}(3,3)$.

$$(6.17)\quad D_3(s,u,n)\bar{J}^{(0)}_{s,u}(3,n) = 3C(s,u+1,w+1)+4C(s+1,u+1,w+1)+4C(s+2,u+1,n-3)$$

$$+ C(s+w+2,u+1,n-w-3)\bar{G}^{(0)}_{s+w+2,u+1}(n-w-3,n-w-3)+C(s+2,u+1,w)\bar{G}^{(0)}_{s+2,u+1}(w,n-3)$$

$$+ C(s+1,u+1,w)\bar{G}^{(0)}_{s+1,u+1}(w,n-2)+C(s,u+1,w+2)\bar{G}^{(0)}_{s,u+1}(w+2,n-1)$$

$$+ D_2(s+w+1,u,n-w-1)\bar{J}^{(0)}_{s+w+1,u}(2,n-w-1) \ .$$

If $n = 2^r + 1$ for some integer $r \geq 4$ (letting $w = 2^{r-2}$)

$$(6.18)\quad D_3(s,u,n)\bar{J}^{(0)}_{s,u}(3,n) = 3C(s,u+1,w+1)+4C(s+1,u+1,w-1)+4C(s+2,u+1,n-3)$$

$$+ C(s+w,u+1,3w)\bar{G}^{(0)}_{s+w,u+1}(3w,3w)+C(s+2,u+1,w-1)\bar{G}^{(0)}_{s+2,u+1}(w-1,n-3)$$

$$+ C(s+1,u+1,w-1)\bar{G}^{(0)}_{s+1,u+1}(w-1,n-2)+C(s,u+1,w)\bar{G}^{(0)}_{s,u+1}(w,n-1)$$

$$+ D_2(s+w,u,n-w)\bar{J}^{(0)}_{s+w,u}(2,n-w) \ .$$

The boundary conditions are for all s,u

$$(6.19)\quad \bar{H}^{(0)}_{s,u}(0) = 0$$

$$(6.20)\quad \bar{G}^{(0)}_{s,u}(1,n) = \bar{H}^{(0)}_{s,u+1}(n-1) \qquad \text{for } n \geq 1.$$

As in the case of $R^{(1)}$, we have attempted to iterate these equations on a computer for $m \leq n \leq 16$ and the uniform a priori density; some results are given in Table III.

The unintegrated expressions $G^{(0)}_{s,u}(m,n)$ and $H^{(0)}_{s,u}(n)$ corresponding to equations (2.4) through (2.7) are not given here. However, the results for $\tilde{R}^{(0)}$ with the uniform a priori density, corresponding to (2.16) and (2.17), are exactly the same as for $R^{(1)}$ since in this case the two procedures are identical for small values of N, and in particular for $N \leq 6$; for $N \geq 13$ the procedures appear to be different in at least one place.

References

[1] Carter, F. L., Jr., 1960. "Group testing in binomial and multinomial situations", Technical Report No. 3, Dept. of Statistics, V.P.I., Blacksburg, Va.

[2] Connor, W. S., 1961. "Group screening designs," Ind. and Engr. Chem., 53, p. 69A.

[3] Curnow, R. N., 1965. "A note on G.S. Watson's paper 'A study of the group screening method'," Technometrics 7, 444-446.

[4] Dorfman, R., 1943. "The detection of defective members of large populations," Ann. Math. Stat. 14, 436-440.

[5] Feller, W., 1957. An Introduction to Probability Theory and Its Applications, 2nd Edition, John Wiley and Sons, New York, p. 225.

[6] Finucan, H.M., 1964. "The blood testing problem," Applied Statistics 13, 43-50.

[7] Kumar, S., 1965. "A group testing problem," Ann. of Math. Statist. 36, 727-728, 1313.

[8] Sobel, M. and Groll, P. A., 1959. "Group testing to eliminate efficiently all defectives in a binomial sample," Bell System Tech. Jour. 38, 1179-1252.

[9] Sobel, M., 1960. "Group testing to classify efficiently all defectives in a binomial sample," a contribution in Information and Decision Processes, Edited by Robert E. Machol, McGraw-Hill, 127-161.

[10] Sobel, M., 1964. "Optimal group testing," Technical Report No. 72, Stanford University, Stanford, Calif.

[11] Sterrett, A., "On the detection of defective members of large populations," Ann. Math. Stat. 28, 1957, 1033-1036.

[12] Ungar, P., 1960. "The cutoff point for group testing," published in Communication on Pure and Applied Math. 13, 49-54.

[13] Watson, G. D., 1961. "A study of the group screening method," Technometrics 3, 371-388.

Expected Number of Tests $H(n|q,R)$ for Procedures $R^{(1)}$, $R^{(2)}$ and the Lower Bound R_1

$n = 6$ **Figure 1.**

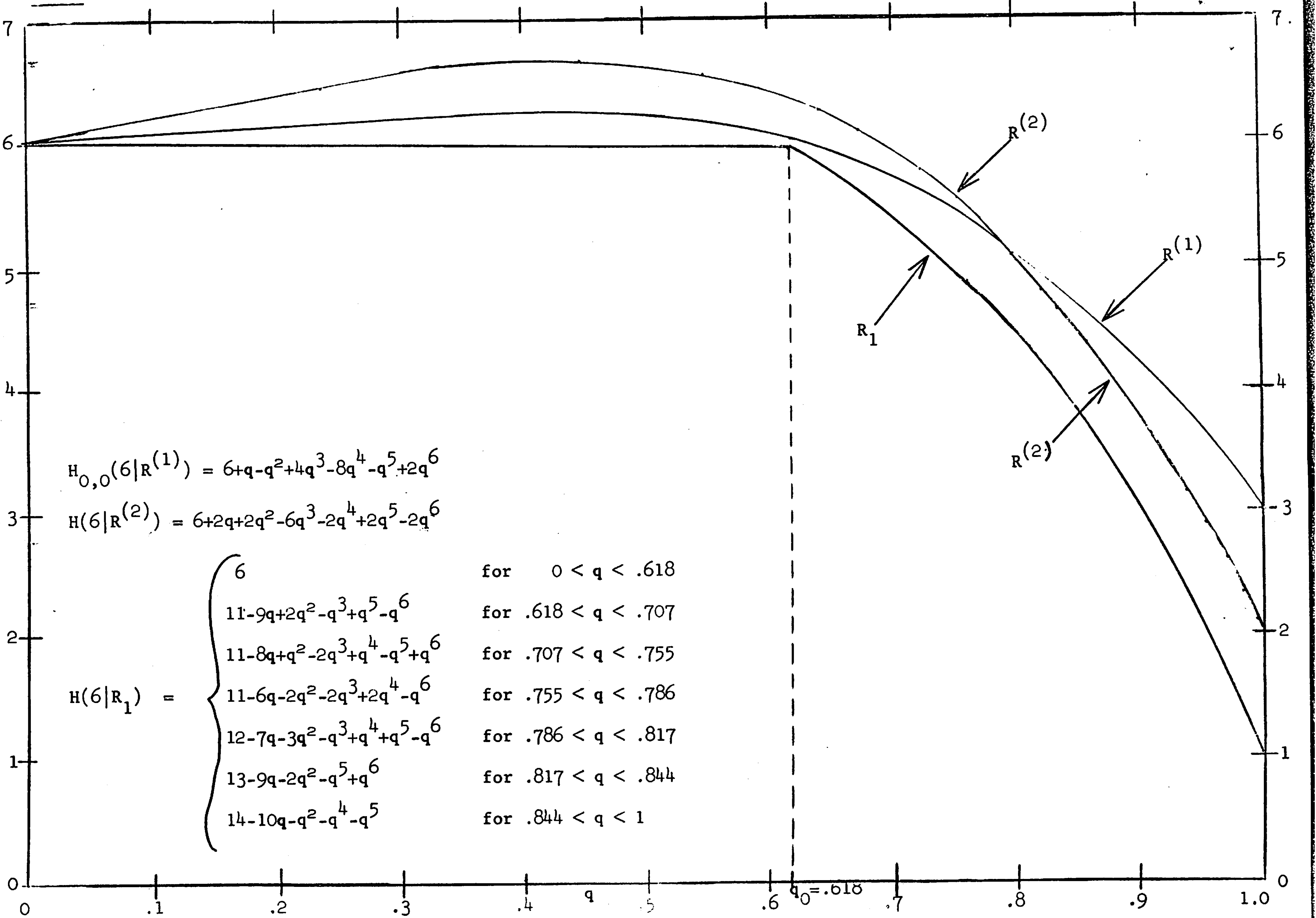
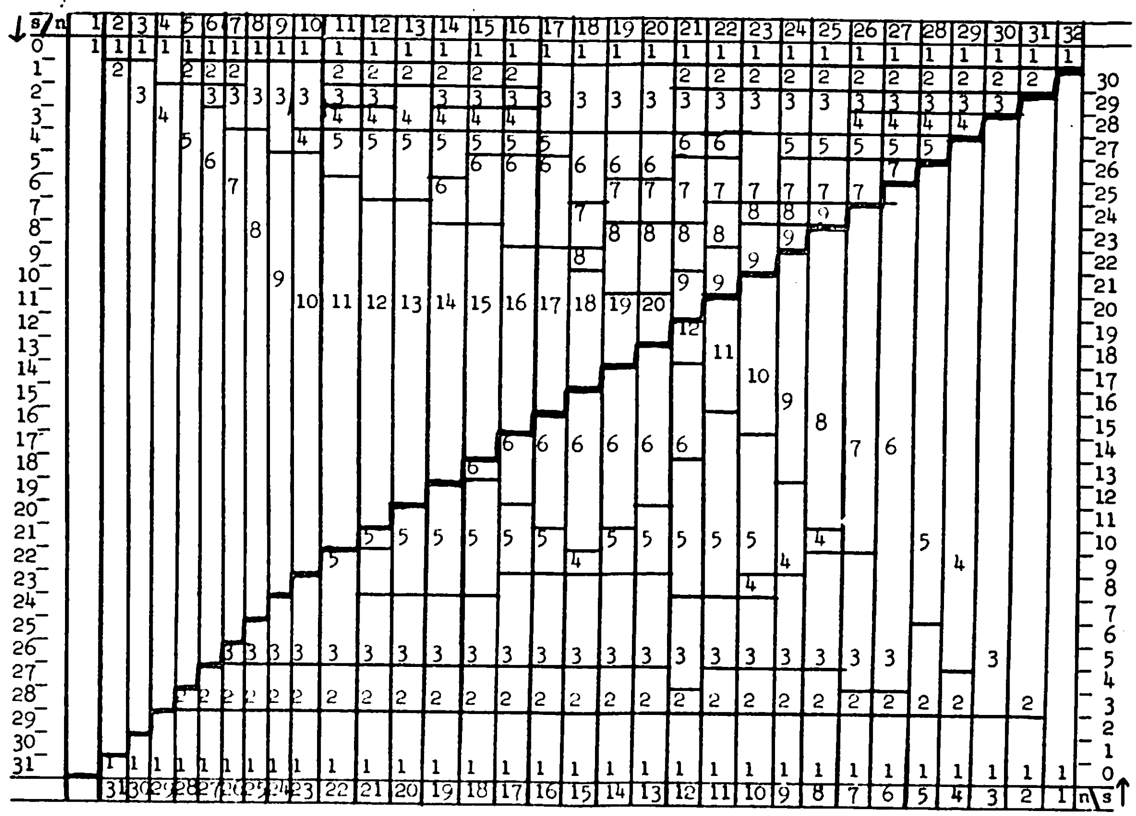


TABLE I

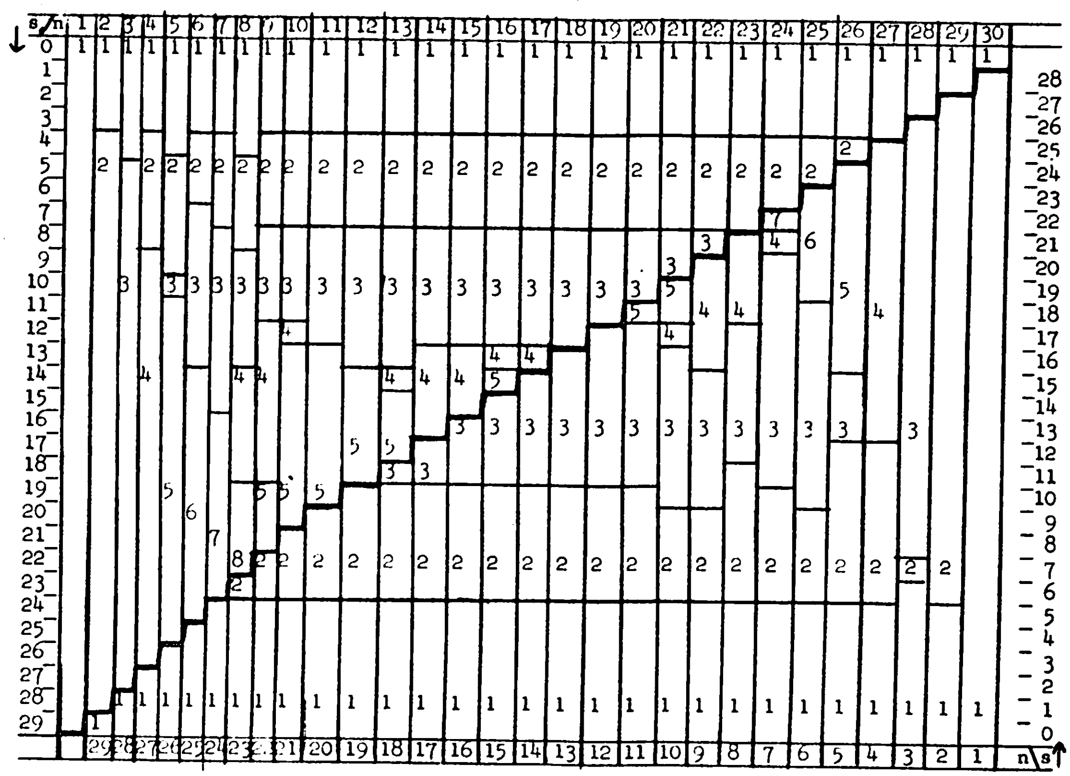
Test Group Sizes $x_H(n, s, u)$ for the H-situation when using the Bayes Procedure $R^{(1)}$ for the Uniform $U(0, 1)$ a Priori Distribution.*

u = 0



u = 1

u = 2

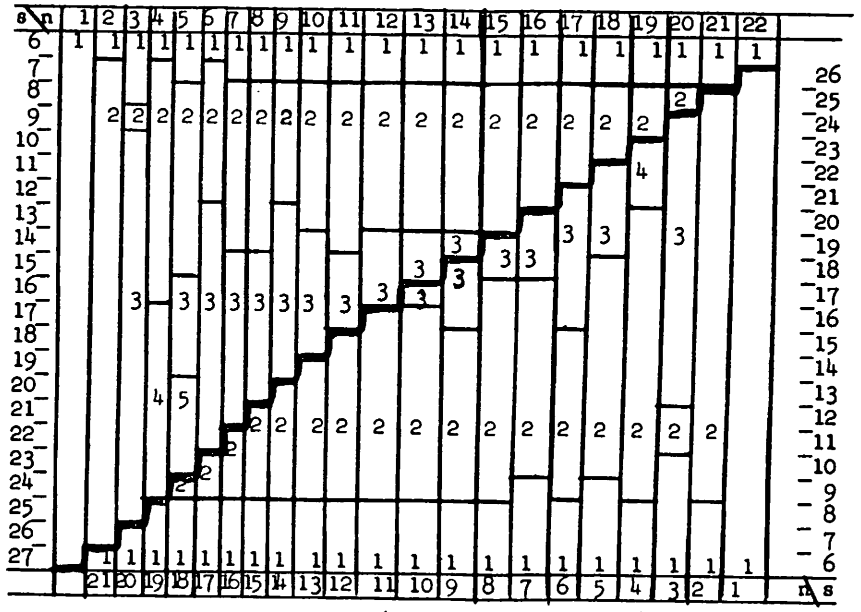


u = 3

*The same table (with an obvious modification) can be used for the Beta $B(\alpha, \beta)$ a priori if α and β are (small) positive integers.

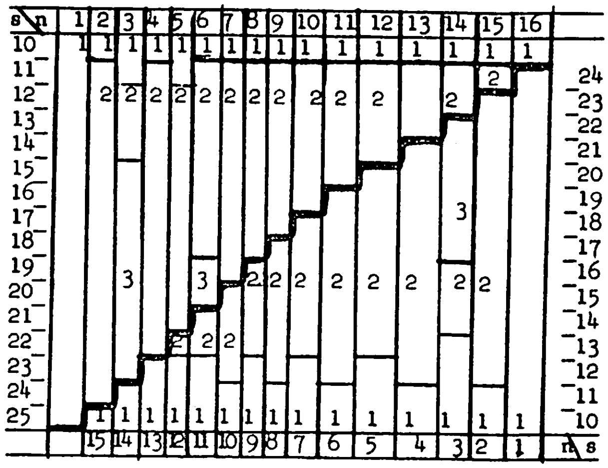
TABLE I CONT.*

u = 4 (Use x_H = 1 for s ≤ 5)



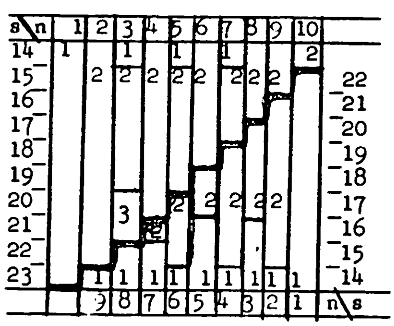
u = 5 (Use x_H = 1 for s ≤ 5)

u = 6 (Use x_H = 1 for s ≤ 9)



u = 7 (Use x_H = 1 for s ≤ 9)

u = 8 (Use x_H = 1 for s ≤ 13)



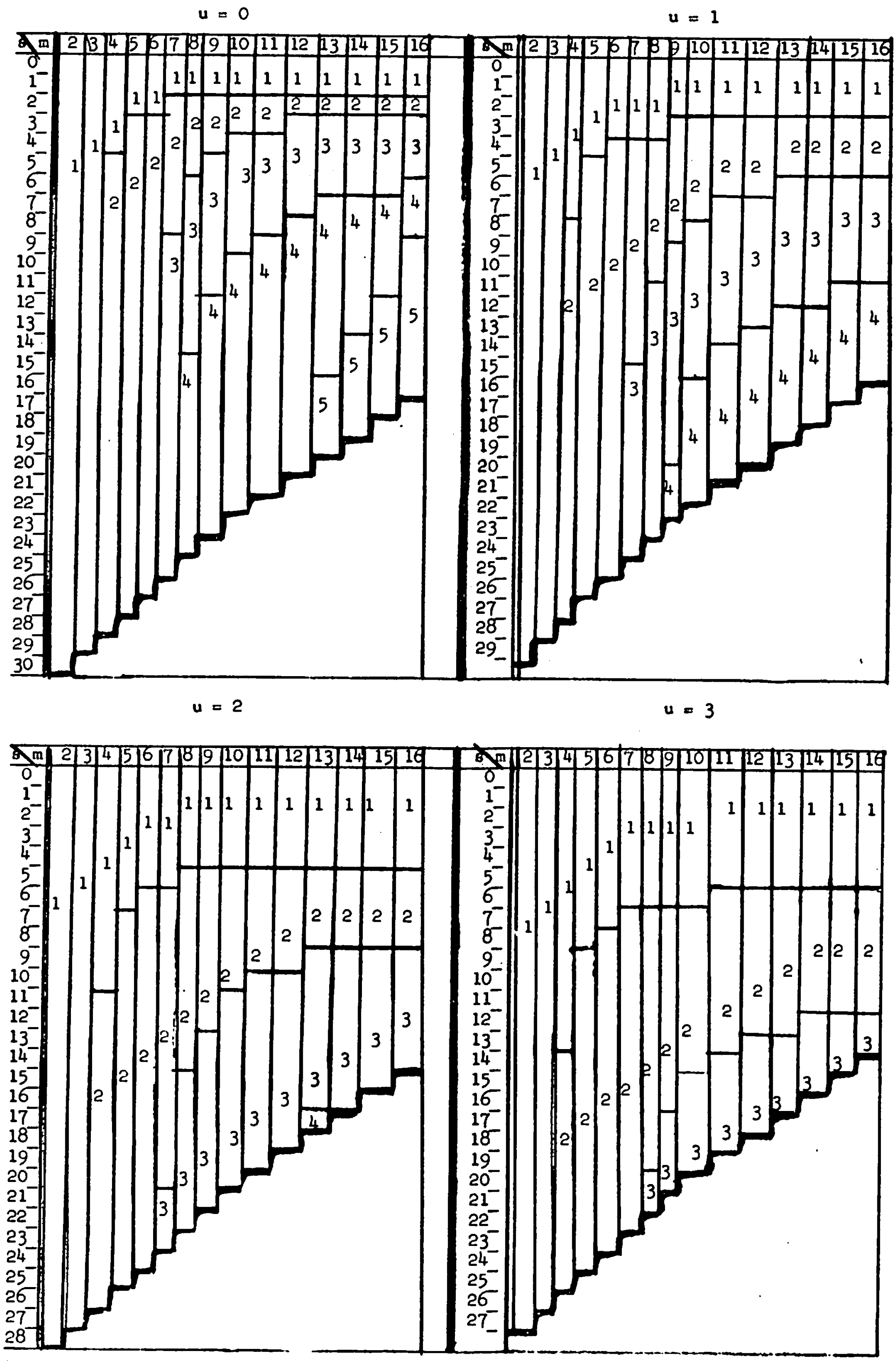
u = 9 (Use x_H = 1 for s ≤ 13)

*$x_H(n, s, u) = 1$ for all n, s, u with $u \geq 10$ and $n + s + u \leq 32$ with the following exceptions:

$x_H(2, 17, 10) = x_H(2, 18, 10) = x_H(2, 19, 10) = x_H(2, 20, 10)$
$= x_H(3, 19, 10) = x_H(4, 17, 10) = x_H(4, 18, 10) = x_H(2, 19, 11) = 2$

TABLE II

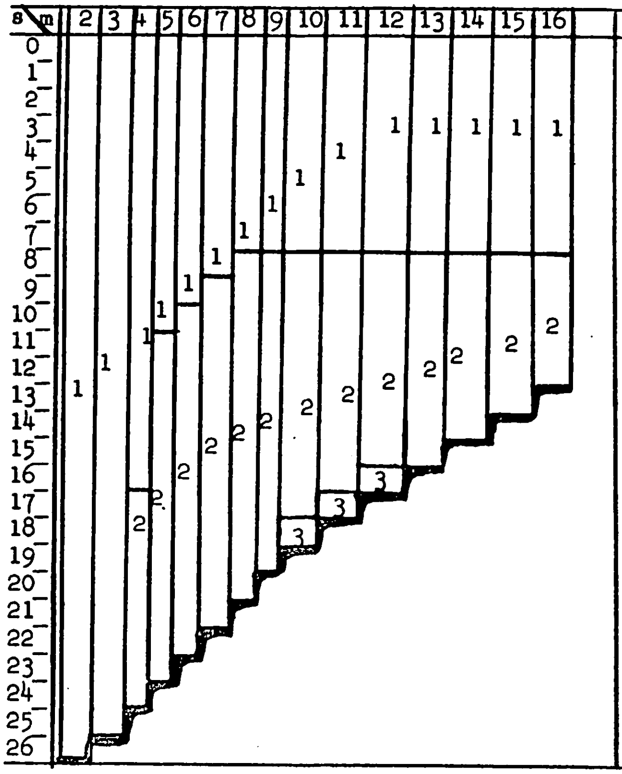
Test Group sizes $x_G(m, s, u)$ for the G-situation when using the Bayes Procedure $R^{(1)}$ for the Uniform U(0, 1) a Priori Distribution.*



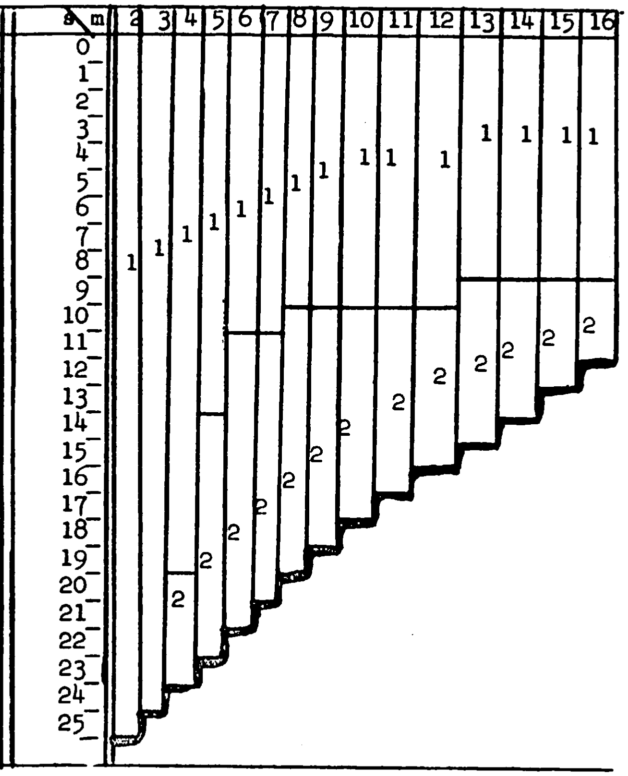
*The same table (with an obvious modification) can be used for the Beta $B(\alpha, \beta)$ a priori if α and β are (small) positive integers.

TABLE II CONT.*

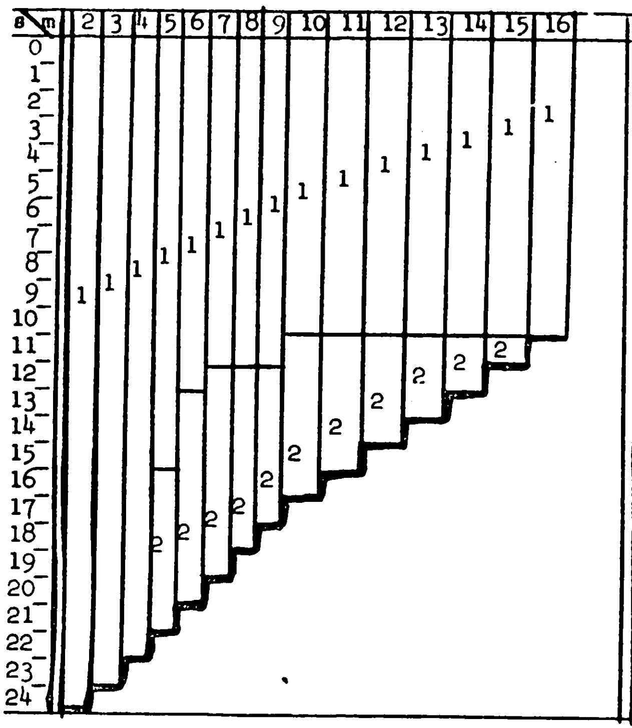
u = 4



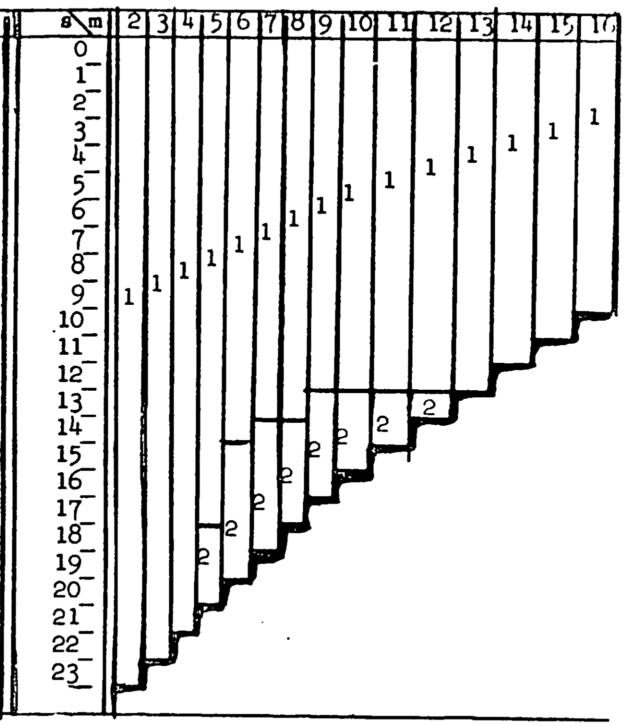
u = 5



u = 6



u = 7



*$x_G(m,s,u) = 1$ for all m, s, u with $u \geq 8$, $m \leq 16$ and $m + s + u \leq 32$ with the following exceptions:

$x_G(6, 17, 8) = x_G(6, 18, 8) = x_G(7, 16, 8) = x_G(7, 17, 8)$

$= x_G(8, 15, 8) = x_G(8, 16, 8) = x_G(9, 15, 8) = 2$

Table III: Bayes Risk and Lower Bounds for Procedures $R^{(1)}$ and $\tilde{R}^{(0)}$ for the Uniform U(0,1) A Priori Distribution.

N	$\bar{H}_{0,0}(N\|R^{(1)})$	$H_{0,0}(N\|\tilde{R}^{(0)})$	LB ① for $\bar{H}_{0,0}(N\|R^{(1)})$	ILB ②
1	1.000	1.000	1.000	0.787
2	2.000	2.000	1.818	1.575
3	2.917	2.917	2.671	2.362
4	3.850	3.850	3.524	3.149
5	4.800	4.800	4.386	3.937
6	5.686	5.686	5.246	4.724
7	6.602	6.602	6.110	5.511
8	7.505	7.505	6.972	6.299
9	8.400	8.400	7.836	7.086
10	9.306	9.306	8.696	7.873
11	10.198	10.198	not calculated	8.661
12	11.089	11.089	"	9.448
13	11.978	11.978	"	10.235
14	12.868	12.868	"	11.023
15	13.757	13.756	"	11.810
16	14.641	14.640	"	12.597
17	15.529	15.528	"	13.384
18	16.414	16.413	"	14.172
19	17.296	17.295	"	14.959
20	18.179	18.178	"	15.746
21	19.061	19.059	"	16.534
22	19.943	19.941	"	17.321
23	20.822	20.821	"	18.108
24	21.701	21.699	"	18.896
25	22.581	not calculated	"	19.683
26	23.459	"	"	20.470
27	24.338	"	"	21.258
28	25.215	"	"	22.045
29	26.093	"	"	22.832
30	26.969	"	"	23.620
31	27.846	"	"	24.407
32	28.722	"	"	25.194
33	29.599	"	"	25.982
34	30.474	"	"	26.769
35	31.349	"	"	27.556
36	32.225	"	"	28.344

① The lower bound LB = $\int_0^1 H(N|R,q)dq$ for $R^{(1)}$ is taken from [8] and cannot be achieved for $n \geq 2$.

② The information theory-lower bound (ILB) is based on (4.3), holds for any group-testing procedure and cannot be achieved for $n \geq 1$.

Table V A

Special Subroutines of Procedure $\tilde{R}^{(0)}$

for $G_{s,u}(2,n)$ and $J_{s,u}(2,n)$ Situations.#

$G_{s,u}(2,2)$:
$$\begin{tikzcd} & S_{s+1,u+1} & S_{s+1,u+1} \\ T(a_1) \arrow[ur] \arrow[r] & T(a_2) \arrow[ur] \arrow[r] & S_{s,u+2} \end{tikzcd}$$

$G_{s,u}(2,n)$: $n \geq 3$
$$\left\{ \begin{array}{l} \begin{tikzcd} & S_{s+n-1,u+1} & S_{s+n-1,u+1} \\ T(a_1,\vec{b}_{n-2}) \arrow[ur] \arrow[r] & T(a_2,\vec{b}_{n-2}) \arrow[ur] \arrow[r] & J_{s,u}(2,n) \end{tikzcd} \\ \begin{tikzcd} & H_{s+1,u+1}(n-2) \\ T(a_1) \arrow[ur] \arrow[r] & H_{s,u+1}(n-1) \end{tikzcd} \end{array} \right.$$

$J_{s,u}(2,3)$:
$$\begin{tikzcd} & S_{s+1,u+2} \\ T(a_1) \arrow[ur] \arrow[r] & G_{s,u+1}(2,2) \end{tikzcd}$$

$J_{s,u}(2,4)$:
$$\begin{tikzcd} & J_{s+1,u}(2,3) \\ T(b_1) \arrow[ur] \arrow[r] & G_{s,u+1}(2,3) \end{tikzcd}$$

If $n \geq 5$ and not of the form $2^r + 2$ for any integer $r \geq 2$

$J_{s,u}(2,n)$:
$$\begin{tikzcd} & G_{s+1,u+1}(n-2,n-2) \\ T(a_1) \arrow[ur] \arrow[r] & G_{s,u+1}(n-1,n-1) \end{tikzcd}$$

If $n = 2^r + 2$ for some integer $r \geq 2$ (letting $t = 2^{r-1}$)

$J_{s,u}(2,n)$:
$$\begin{tikzcd} & J_{s+t,u}(2,t+2) \arrow[r] & G_{s+1,u}(t,n-2) \\ T(\vec{b}_t) \arrow[ur] \arrow[r] & T(a_1) \arrow[ur] \arrow[r] & G_{s,u+1}(t,n-1) \end{tikzcd}$$

The symbol T( ) indicates a test on the units shown in the parens. A horizontal (slanted) arrow indicates a failed (passed) test. The symbol $S_{s,u}$ denotes a terminal stop with s satisfactory and u unsatisfactory units. The symbols $G_{s,u}(m,n)$, $H_{s,u}(n)$ and $J_{s,u}(m,n)$ indicate a posteriori situations. In a G-situation, the a_i (explanation continued on next page)

Table V B

Special Subroutines of Procedure $\tilde{R}^{(0)}$ for $G_{s,u}(3,n)$ and $J_{s,u}(3,n)$ Situations.#

$G_{s,u}(3,n)$:

$$\begin{tikzcd}
 & S_{s+n-1,u+1} & S_{s+n-1,u+1} & S_{s+n-1,u+1} \\
T(a_1,a_2,\vec{b}_{n-3}) \arrow[ur] \arrow[r] & T(a_1,a_3,\vec{b}_{n-3}) \arrow[ur] \arrow[r] & T(a_2,a_3,\vec{b}_{n-3}) \arrow[ur] \arrow[r] & J_{s,u}(3,n)
\end{tikzcd}$$

$$\begin{tikzcd}
 & G_{s+1,u}(2,n-1) \\
T(a_1) \arrow[ur] \arrow[r] & H_{s,u+1}(n-1)
\end{tikzcd}$$

$J_{s,u}(3,3)$:

$$\begin{tikzcd}
 & S_{s+1,u+2} \\
T(a_1) \arrow[ur] \arrow[r] & G_{s,u+1}(2,2)
\end{tikzcd}$$

(Same as $J_{s,u}(2,3)$)

$J_{s,u}(3,4)$:

$$\begin{tikzcd}
 & J_{s+1,u}(2,3) \\
T(b_1) \arrow[ur] \arrow[r] & G_{s,u+1}(3,3)
\end{tikzcd}$$

$J_{s,u}(3,5)$:

$$\begin{tikzcd}
 & J_{s+1,u}(2,4) \\
T(a_1) \arrow[ur] \arrow[r] & G_{s,u+1}(4,4)
\end{tikzcd}$$

$J_{s,u}(3,6)$:

$$\begin{tikzcd}
 & J_{s+2,u}(3,4) & G_{s+1,u+1}(3,4) \\
T(b_1,b_2) \arrow[ur] \arrow[r] & T(b_1) \arrow[ur] \arrow[r] & G_{s,u+1}(3,5)
\end{tikzcd}$$

$J_{s,u}(3,n)$:
$n=7,8,9$

$$\begin{tikzcd}
 & J_{s+2,u}(2,n-2) & G_{s+2,u+1}(n-3,n-3) & G_{s+1,u+1}(2,n-2) \\
T(a_1,b_1) \arrow[ur] \arrow[r] & T(a_2,a_3) \arrow[ur] \arrow[r] & T(a_1) \arrow[ur] \arrow[r] & G_{s,u+1}(2,n-1)
\end{tikzcd}$$

(cont. previous page) are units in the defective set and the b_i are units in the binomial set; let $\vec{b}_r = (b_1,\dots,b_r)$ denote the first r of the b's. In a J-situation, we keep the same notation for the units, although the b's are no longer associated with independent binomial chance variables.

If $n = 2^r + 2$ for some integer $r \geq 3$ (letting $t = 2^{r-1}$, $t' = t/2$, $t'' = t/4$)

$$\begin{tikzcd}[column sep=small]
 & & & & & & G_{s+t-3,u+1}(3,t+2) \\
 & & & & & T(b_{t-1}) \arrow[r] \arrow[ur] & G_{s+t-4,u+1}(3,t+3) \\
 & & & T(b_{t'+1},\ldots,b_{3t''}) & \cdots & \vdots & \vdots \\
 & & & & & & G_{s+3,u+1}(3,n-4) \\
 & & & & & T(b_3) \arrow[r] \arrow[ur] & G_{s+2,u+1}(3,n-3) \\
 & & J_{s+t,u}(3,t+2) & & & & G_{s+1,u+1}(3,n-2) \\
J_{s,u}(3,n): & T(\vec{b}_t) \arrow[r] \arrow[ur] & T(\vec{b}_{t'}) \arrow[r] \arrow[uuuur] & T(\vec{b}_{t''}) & \cdots & T(b_1) \arrow[r] \arrow[ur] & G_{s,u+1}(3,n-1)
\end{tikzcd}$$

For example, for $n = 10$ (i.e., $r = 3$) we obtain

$$\begin{tikzcd}[column sep=small]
 & & & & G_{s+3,u+1}(3,6) \\
 & & & T(b_3) \arrow[r] \arrow[ur] & G_{s+2,u+1}(3,7) \\
 & & J_{s+4,u}(3,6) & & G_{s+1,u+1}(3,8) \\
J_{s,u}(3,10): & T(\vec{b}_4) \arrow[r] \arrow[ur] & T(b_1,b_2) \arrow[r] \arrow[uur] & T(b_1) \arrow[r] \arrow[ur] & G_{s,u+1}(3,9)
\end{tikzcd}$$

If $2^r + 3 \leq n \leq 2^{r+1}$ for some integer $r \geq 3$ (letting $w = 2^{r-2}$)

$$\begin{tikzcd}[column sep=small]
 & & & & & G_{s+2,u+1}(w,n-3) \\
 & & J_{s+w+1,u}(2,n-1-w) & G_{s+w+2,u+1}(n-w-3,n-w-3) & T(a_2) \arrow[r] \arrow[ur] & G_{s+1,u+1}(w,n-2) \\
J_{s,u}(3,n): & T(a_1,\vec{b}_w) \arrow[r] \arrow[ur] & T(a_2,a_3,\vec{b}_w) \arrow[r] \arrow[ur] & T(a_1) \arrow[r] \arrow[ur] & G_{s,u+1}(w+2,n-1) &
\end{tikzcd}$$

If $n = 2^r + 1$ for some integer $r \geq 4$ (letting $w = 2^{r-2}$

$$
\begin{tikzcd}[column sep=small]
 & & & & & G_{s+2,u+1}(w-1,n-3) \\
 & & J_{s+w,u}(2,3w+1) & G_{s+w,u+1}(3w,3w) & T(a_2) \arrow[ur] \arrow[r] & G_{s+1,u+1}(w-1,n-2) \\
J_{s,u}(3,n): & T(a_1,\vec{b}_{w-1}) \arrow[ur] \arrow[r] & T(a_2,a_3,\vec{b}_{w-2}) \arrow[ur] \arrow[r] & T(a_1) \arrow[ur] \arrow[r] & G_{s,u+1}(w,n-1) &
\end{tikzcd}
$$